Nearly Geodesic Riemannian Cubics in $SO(3)$

Lyle Noakes

School of Mathematics and Statistics
The University of Western Australia
Nedlands, WA 6009
Perth, AUSTRALIA
lyle@maths.uwa.edu.au

Abstract: *Riemannian cubics* are curves in a manifold M that satisfy a variational condition appropriate for interpolation problems. When M is the rotation group $SO(3)$, Riemannian cubics are track-summands of *Riemannian cubic splines*, used for motion planning of rigid bodies. Partial integrability results are known for Riemannian cubics, and the asymptotics of Riemannian cubics in $SO(3)$ are reasonably well understood. The mathematical properties and medium-term behaviour of Riemannian cubics in $SO(3)$ are known to be be extremely rich, but there are numerical methods for calculating Riemannian cubic splines in practice. What is missing is an understanding of the short-term behaviour of Riemannian cubics, and it is this that is important for applications. The present paper fills this gap by deriving approximations to nearly geodesic Riemannian cubics in terms of elementary functions. The high quality of these approximations depends on mathematical results that are specific to Riemannian cubics.

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1 Introduction

Suppose that a C^∞ curve $x : \mathbb{R} \to M$ in a Riemannian manifold M is sampled at times $t_0 < t_1 < \ldots < t_n$, yielding observations $x(t_i) = x_i$ for $0 \leq i \leq n$. Then x is uniformly approximated by a track-sum of minimal geodesic arcs joining successive observations, with $O(\delta^2)$ error, where δ is the maximum distance between x_{i-1} and x_i, for $1 \leq i \leq n$. So any C^∞ curve in M is a track-sum of curves that are nearly geodesic. Although x is C^∞, the piecewise-geodesic approximation usually fails to be C^1 at junctions.

A C^2 approximation is given by a *natural Riemannian cubic spline*, namely a track-sum of *Riemannian cubics*, critical for the mean squared norm of covariant acceleration. If sampling is sufficiently frequent we can restrict attention to Riemannian cubics that are *nearly geodesic*.

In the special case where M is flat, Riemannian cubics are expressed simply in terms of cubic polynomials. When M is curved, Riemannian cubics are given by a differential equation that is difficult to solve, even when M is the unit 3-sphere with the standard metric, or the bi-invariant rotation group $SO(3)$. These cases occur in motion planning for rigid bodies, and so numerical methods are needed to find Riemannian cubic splines [6].

The asymptotics of Riemannian cubics are studied in [10], [11], [14], but little is known about the short-term behaviour of cubics, and the short term behaviour is more relevant for motion planning of rigid bodies. The present paper fills this gap, by deriving approximations to nearly geodesic Riemannan cubics in terms of trigonometric functions and polynomials. The new approximations are much more informative than Taylor approximation, either in coordinate charts or ambient space, and capture interesting short term geometry of Riemannian cubics that was previously observed in numerical experiments.

Before describing the methods and layout of the present paper, we review Riemannian cubics in more detail.

2 Riemannian cubics

For M a finite-dimensional Riemannian manifold, consider the functional

$$J(x) \ := \ \int_{t_0}^{t_1} \langle \nabla_t x^{(1)}(t), \nabla_t x^{(1)}(t) \rangle \, dt$$

defined on C^∞ curves $x : [t_0, t_1] \to M$, where x and its derivative $x^{(1)}$ are prescribed at t_0, t_1. Here ∇ denotes the Levi-Civita covariant derivative defined by the Riemannian metric $\langle \ , \ \rangle$. A *Riemannian cubic* is a critical point x of J, in the sense that x satisfies the associated 4th order Euler-Lagrange equation [9]

$$\nabla_t^3 x^{(1)} + R_{x(t)}(\nabla_t x^{(1)}, x^{(1)}) x^{(1)} \ = \ \mathbf{0} \qquad (1)$$

where R is the Riemannian curvature. The existence of Riemannian cubics satisfying the prescribed conditons is proved in [5] when M is a complete Riemannian manifold. As seen from (1), cubically reparameterised geodesics are Riemannian cubics. However most Riemannian cubics do not arise in this way.

Definition 1 *Given $\epsilon > 0$, a C^∞ curve $x : [t_0, t_1] \to M$ is (ϵ-)*nearly geodesic *when, for all $t \in [t_0, t_1]$,*

$$\|\nabla_t x^{(1)}(t)\| \ < \ \epsilon \quad \textit{and} \quad \|\nabla_t^2 x^{(1)}(t)\| \ < \ \epsilon.$$

□

Riemannian cubics need not be nearly geodesic, but the restriction of any C^∞ curve to a sufficiently small subinterval can be reparameterized to a nearly geodesic curve defined over a fixed interval $[t_0, t_1]$. So nearly geodesic curves are informative about the local geometry of arbitrary C^∞ curves, in particular Riemannian cubics. Nearly geodesic curves arise naturally in other ways too.

Example 1 *Take M to be the matrix group $SO(3)$ of rotations of Euclidean 3-space E^3. Define $x(t) \in SO(3)$ by taking its columns to be the coordinates at time t of an orthonormal frame fixed relative to some rigid body B. If the mass distribution of B is spherically symmetric, and if B moves freely, then x is a geodesic in $SO(3)$ with*

respect to a bi-invariant Riemannian metric. If, however, B is subject to a C^1-uniformly small torque T then x is nearly geodesic.

Let T be unknown, and suppose that the configuration of B and its angular velocity are observed at times t_0, t_1. Then a minimiser x of J is an interpolant that minimises the mean-squared torque. Since T is C^1-uniformly small, the interpolant x is a nearly geodesic Riemannian cubic. □

We refer to [2, 14, 15, 3, 4] for further applications of Riemannian cubics.

When M is Euclidean m-space E^m, a Riemannian cubic is precisely a polynomial curve of degree ≤ 3. Nothing like this can be said for non-flat manifolds M, even when M is a space of constant nonzero curvature. The situation for *elastic curves* is entirely different.

Example 2 *An* elastic curve *in M is a critical point of the restriction of J to the space of* constant-speed *curves x, with x and $x^{(1)}$ still prescribed at t_0, t_1. When $M = E^3$ elastic curves are the Euler elastica, whose curvature and torsion are obtained in terms of elliptic sine function, as in Lecture 1 of* [18]*, whereas Riemannian cubics in E^3 are just cubic polynomial curves.*

On the other hand, when M is the unit sphere S^3 in E^4, there are quadrature formulae for elastic curves in terms of elliptic functions [8, 1, 17] *and Lecture 2 of* [18]*, but quadrature formulae for Riemannian cubics in S^3 are known only for a codimension* 3 *subclass* [16]*. Elastic curves do not resemble Riemannian cubics, except for the very short term.* □

Whereas the long term behaviour of Riemannian cubics is studied in [10], [12], [11], their short and medium term behaviour is poorly understood. Yet the short and medium term are more significant in applications, such as interpolation and motion planning for rigid bodies.

Example 3 *Given $t_0 < t_1 < \ldots < t_n$ and $x_0, x_1, \ldots, x_n \in M$ define*

$$J(x) \;:=\; \int_{t_0}^{t_n} \langle \nabla_t x^{(1)}(t), \nabla_t x^{(1)}(t) \rangle \; dt$$

defined on C^2 curves $x : [t_0, t_n] \to M$ satisfying $x(t_i) = x_i$ for $i = 0, 1, \ldots, n$. A critical point of J is called a natural cubic spline*. Natural cubic splines are characterised as C^2 track sums of Riemannian cubics on the intervals $[t_{i-1}, t_i]$ for $i = 1, 2, \ldots, n$, whose covariant acceleration vanishes at t_0 and t_n.* □

As we shall see in Example 9, Taylor approximations are of limited value in this context. Much more accurate estimates can be made by exploiting specific properties of Riemannian cubics x, especially when x is nearly geodesic.

We focus on bi-invariant $M = SO(3)$ of rotations of E^3, and on *Lie quadratics* namely the *left Lie reductions* $V : [t_0, t_1] \to E^3$ of a Riemannian cubics x. The Lie quadratic V_δ of a nearly geodesic Riemannian cubic x_δ is nearly constant. The variational equations of Lie quadratics are used to find approximations of V_δ. There is a quadrature formulae of [12] for Riemannian cubics in terms of Lie quadratics, but the approximate Lie quadratics cannot be directly substituted for V_δ into the formula. Nonetheless, taking a little more care, we obtain an approximation $\hat{x}$ to x_δ.

In effect, the known structural results for Riemannian cubics in $SO(3)$ are exhausted, before resorting to a Taylor approximation in what is left. Surprisingly, whereas the reconstruction of Riemannian cubics from Lie quadratics requires a quadrature [12], the first order approximation $\hat{x}$ for x_δ given in Theorem 4 is algebraic[1] in trigonometric functions and low degree polynomials.

The layout of the paper is as follows.

- §3 is an introduction to Lie reductions of Riemannian cubics in bi-invariant Lie groups G.
- §4 studies the variational equation of a Lie quadratic, giving examples where the variational equation can be solved exactly. Example 6 concerns the simplest case of variations through nearly constant Lie quadratics. Derivatives $V^{(i,0)}$ to order $0 \leq i \leq n$ of a variation to order n with respect to the variation parameter, give rise to an order n approximation $\hat{V}_n$ of a nearly constant Lie quadratic V_δ.

[1] The expression is complicated, but it is difficult to see any way around this.

- In §5, $\mathcal{G}$ is taken to be the Lie algebra $so(3) \cong E^3$ of the rotation group $SO(3)$. Theorems 2, 3 give formulae for the first and second order approximations $\hat{V}_1$ and $\hat{V}_2$ to the Lie quadratic V_δ. In applications such as Example 1, V_δ gives the angular momentum relative to the body B.

  Already $\hat{V}_1$ captures significant medium term behaviour of V_δ, as illustrated in Figures 1, 2 of Example 9. Taylor approximations to V_δ perform badly (Figure 1), while $\hat{V}_1$ and $\hat{V}_2$ are nearly indistinguishable from V_δ in the medium term. For the longer term, $\hat{V}_2$ significantly improves on $\hat{V}_1$ (Figure 2).

- In §6 the approximations $\hat{V}_n$ for $n = 1$ and $n = 2$ are used in combination with Theorem 5 of [12] to derive a first order approximation $\hat{x}$ to the Riemannian cubic x_δ, namely $x_\delta^{(j)}(t) = \hat{x}^{(j)}(t) + O(\delta^2)$ for any non-negative integer j. This is our main result, stated as Theorem 4, and implemented in Example 10.

3 Riemannian cubics in Bi-Invariant Lie groups: Lie quadratics

Now we take M to be a path-connected finite-dimensional Lie group G, with bi-invariant pseudo-Riemannian metric. The restriction of the metric to the Lie algebra $\mathcal{G}$ is an ad-*invariant* semi-definite inner product, namely $\mathrm{ad}(u) : \mathcal{G} \to \mathcal{G}$ is skew-adjoint for all $u \in \mathcal{G}$. Conversely an ad-invariant semi-definite inner product[2] on $\mathcal{G}$ extends by left multiplication to a bi-invariant pseudo-Riemannian metric on G. Define the *left Lie reduction* V of $x : [t_0, t_1] \to G$ by

$$V(t) := dL(x(t)^{-1})_{x(t)}(x^{(1)}(t)$$

where $L(g)$ denotes left multiplication by $g \in G$.

Theorem 1 ([9, 10]) *x is a Riemannian cubic in G if and only if, for all $t \in [t_0, t_1]$, and some $C \in \mathcal{G}$, we have*

$$V^{(2)}(t) \;=\; [V^{(1)}(t), V(t)] + C. \tag{2}$$

□

Equation (2) is second order, whereas the Euler-Lagrange equation (1) has order 4. Equivalently we may write

$$V^{(3)}(t) \;=\; [V^{(2)}(t), V(t)]. \tag{3}$$

A curve $V : [t_0, t_1] \to \mathcal{G}$ satisfying (2) for all t is said to be a *Lie quadratic*. The Lie quadratic and Riemannian cubic are said to be *null* when $C = \mathbf{0}$, and *non-null* otherwise. Null Lie quadratics in E^3 appear in applications in fluid dynamics [7].

If $x : [t_0, t_1] \to G$ is a Riemannian cubic then so are x^{-1} and gx where $g \in G$. Geodesics in bi-invariant Lie groups are precisely the Riemannian cubics with constant Lie quadratics.

Example 4 *An affine line in $\mathcal{G}$ is a Lie quadratic, and is null when its image contains $\mathbf{0}$. A Lie algebra $\mathcal{G}$ is abelian precisely when all its affine lines are null Lie quadratics. Then all nonconstant null Lie quadratics are affine lines, and a Lie quadratic is precisely a polynomial curve in $\mathcal{G}$ of degree at most 2.* □

From Theorem 1 a short calculation shows

Corollary 1 *Let $y, z : [t_0, t_1] \to G$ be restrictions of 1-parameter subgroups of G, with infinitesimal generators $A, B \in \mathcal{G}$. Then the pointwise product $t \mapsto y(t)z(t)$ is a Riemannian cubic if and only if $[[[B, A], A], B] = \mathbf{0}$.* □

Example 5 *Let G be bi-invariant $SO(3)$, with $A, B \in so(3)$ linearly independent. Then the pointwise product $t \mapsto \exp(tA)\exp(tB)$ is a Riemannian cubic if and only if $\langle A, B \rangle = 0$.* □

Null Lie quadratics in E^3 are studied in [10, 14]. The present paper focuses on the *non-null* Lie quadratics and Riemannian cubics, which are generic, and whose behaviour can be complicated, as illustrated in [11].

[2] These do not exist for some Lie groups. However for G semisimple we may use the Killing form.

4 Variational Equations For Lie Quadratics

For $t_0 < t_1$ and any C^∞ map $V : [-1,1] \times [t_0, t_1] \to \mathcal{G}$, define $V_h : [t_0, t_1] \to \mathcal{G}$ by $V_h(t) = V(h,t)$, where $h \in [-1,1]$. Denote the i-fold derivative of V with respect to h of the j-fold derivative with respect to t, by $V^{(i,j)}$. Suppose that, for each $h \in [-1,1]$, V_h is a Lie quadratic. Differentiating $V^{(0,3)} = [V^{(0,2)}, V^{(0,0)}]$ $n \geq 1$ times with respect to h, we obtain the third order linear ODE for $Y_n(t) := V^{(n,0)}(h,t)$

$$Y_n^{(3)} + \mathrm{ad}(V^{(0,0)})Y_n^{(2)} - \mathrm{ad}(V^{(0,2)})Y_n \;=\; \sum_{i=1}^{n-1} \left(\begin{array}{c} n \\ i \end{array} \right) [V^{(n-i,2)}, V^{(i,0)}] \tag{4}$$

where the coefficients and right hand side are in terms of derivatives with respect to t of Y_i with $i = 0, 1, 2, \ldots, n-1$.

Let V_0 be the Lie quadratic of a nontrivial cubically reparameterised geodesic, namely $V_0 = q_0(t)D$ for $\mathbf{0} \neq D \in \mathcal{G}$ and $q_0 : \mathbb{R} \to \mathbb{R}$ is a quadratic polynomial. For $n = 1$, the variational equation (4) evaluated at $h = 0$ becomes

$$Y_1^{(3)}(t) + \mathrm{ad}(D)(q_0(t)Y_1^{(2)}(t) - q_0^{(2)}Y_1(t)) \;=\; \mathbf{0}. \tag{5}$$

Let $\mathrm{ad}(D) : \mathcal{G} \to \mathcal{G}$ be diagonalizable[3] over $\mathbb{C}$, namely $\mathcal{G} = \mathcal{K} + \mathcal{E}$ where $\mathcal{K}$ is the kernel of $\mathrm{ad}(D)$, and $\mathcal{E}$ has a basis of eigenvectors E_i of $\mathrm{ad}(D)$. For $K \in \mathcal{K}$, (5) gives

$$\langle Y_1^{(3)}(t), K \rangle = -\langle \mathrm{ad}(D)(q_0(t)Y_1^{(2)}(t) - q_0^{(2)}Y_1(t)), K \rangle = \langle q_0(t)Y_1^{(2)}(t) - q_0^{(2)}Y_1(t), \mathrm{ad}(D)K \rangle = 0.$$

So $\langle Y_1(t), K \rangle = q_K(t)$ where $q_K : [t_0, t_1] \to \mathbb{R}$ is another quadratic polynomial. Similarly, $\langle Y_1(t), E_i \rangle$ satisfies

$$y^{(3)}(t) \;=\; r(t)y^{(2)}(t) - r^{(2)}y(t) \implies y^{(2)}(t) \;=\; r(t)y^{(1)}(t) - r^{(1)}(t)y(t) + c_0 \tag{6}$$

where $r(t) := \lambda_i q_0(t)$, $\lambda_i \in \mathbb{C}$ is the eigenvalue of of E_i, and c_0 is constant. The linear ODE (6) is solvable by quadratures in terms of solutions of the associated homogeneous ODE

$$y^{(2)}(t) \;=\; r(t)y^{(1)}(t) - r^{(1)}(t)y(t). \tag{7}$$

Example 6 *Let r be a constant $0 \neq k \in \mathbb{C}$. Then (7) reads $y^{(2)}(t) = ky^{(1)}(t) \implies y(t) = c_2 + c_1 e^{kt}$ where c_1, c_2 are constant. So the general solution of (6) is*

$$y(t) \;=\; -\frac{c_0}{k}t + c_2 + c_1 e^{kt}.$$

□

Example 7 *Let r be nonconstant and linear. After a time-shift, write $r(t) = kt$ where $0 \neq k \in \mathbb{C}$ is constant. Then (7) becomes $y^{(2)}(t) = kty^{(1)}(t) - ky(t)$ and, substituting $y(t) = tz(t)$,*

$$2z^{(1)} + tz^{(2)} \;=\; kt^2tz^{(1)} \implies z^{(1)}(t) \;=\; c_2 t^{-2} e^{kt^2/2}$$

from which $z(t)$ is found by quadrature. □

Example 8 *Let $r = kt^2$ where k is constant. Equation (7) becomes $y^{(2)}(t) = kt^2y^{(1)}(t) - 2kty(t)$. Substituting $y(t) = w(s)$ with $s = kt^3/3$, gives a form of Kummer's equation*

$$sw'' \;=\; (s - \frac{2}{3})w' + \frac{2}{3}w$$

where w' and w'' are derivatives with respect to s. So w is given in terms of confluent hypergeometric functions. □

[3] In a semisimple Lie algebra [19], the so-called *semisimple* elements D for which this holds comprise an open dense subset of $\mathcal{G}$.

Suppose now that V_0 is constant, and let $h \mapsto V(h,0)$ be an affine function of uniform norm ≤ 1. From equation (6), for any non-negative integers n and j, for some some $c_{n,j} > 0$, and for all $(h,t) \in [-1,1] \times [t_0,t_1]$,

$$\|V^{(n,j)}(h,t)\| \ < \ c_{n,j}.$$

For $\delta \in (0,1)$ and $n \geq 1$, the *nth order approximate quadratic* $\hat{V}_n : [t_0,t_1] \to \mathcal{G}$ of $V : [-1,1] \times [t_0,t_1] \to \mathcal{G}$ is defined by

$$\hat{V}_n(t) \ := \ V_0(t) + \sum_{i=1}^{n} \frac{\delta^i V^{(i,0)}(0,t)}{i!}.$$

By Taylor's Theorem, $\hat{V}_n^{(j)}(t) = V_\delta^{(j)}(t) + O(\delta^{n+1})$ where the asymptotic constants depend on V_0, n and j. Once $\mathcal{G}$ is specified the analysis of Example 6 can be taken further.

5 Approximating Nearly Constant Lie Quadratics in $so(3)$

Considering Euclidean 3-space E^3 as a Lie algebra, with respect to the cross-product as Lie-bracket, the Euclidean inner product $\langle\ ,\ \rangle$ is ad-invariant. The Lie isomorphism $\mathrm{ad} : E^3 \to so(3)$, given by $\mathrm{ad}(v)(w) := v \times w$, identifies E^3 with $so(3)$. In particular, an ad-invariant inner product $\langle\ ,\ \rangle$ on $so(3)$ is defined by requiring ad to be an isometry. Take $G = SO(3)$ with the corresponding bi-invariant Riemannian metric.

We seek nth order approximate quadratics $\hat{V}_n : [t_0,t_1] \to so(3)$ of variations V, where V_0 is a nonzero constant $D \in so(3)$ and $h \mapsto V(h,0)$ is affine. Then $\mathrm{ad}(D)$ is diagonalizable, the linear span $\mathcal{H}$ of D is a Cartan subalgebra, and $\mathrm{ad}(D) : so(3) \otimes_{\mathbb{R}} \mathbb{C} \to so(3) \otimes_{\mathbb{R}} \mathbb{C}$ has eigenvalues $\pm d\mathbf{i}$, with unit eigenvectors $E_1, E_2 = \bar{E}_1 \in so(3) \otimes_{\mathbb{R}} \mathbb{C}$ orthogonal to D. Define unit vectors

$$F_0 \ := D/d, \quad F_1 \ := \ (E_1 + E_2)/\sqrt{2}, \quad F_2 \ := \ -(E_1 - E_2)/(\sqrt{2}\mathbf{i})$$

with $[F_0, F_1] = F_2$ and $[F_2, F_0] = F_1$. Then F_0, F_1, F_2 corresponds under ad to a positively oriented orthonormal basis of E^3.

For $n \geq 1$ define $f_n : [t_0,t_1] \to \mathbb{R}$ and $v_n : [t_0,t_1] \to F_0^\perp$ by $f_n(t)F_0 + v_n(t) := Y_n(t)$. Taking $k = \pm d\mathbf{i}$ in Example 6, f_1 is a quadratic polynomial q, and $v_1 = A + \tilde{e}(B)$ where $A : [t_0,t_1] \to F_0^\perp \subset so(3)$ is affine, $B \in so(3)$, and $\tilde{e}(t)$ is the Lie endomorphism $\exp(-d(t-t_0)\mathrm{ad}(F_0))$ of $so(3)$. We have proved

Theorem 2 *The first order approximate quadratic of V has the form*

$$\hat{V}_1(t) \ = \ D + \delta(q(t)F_0 + A_0 + (t-t_0)A_1 + \tilde{e}(t)(B))$$

where $q(t) = c_0 + c_1(t-t_0) + c_2(t-t_0)^2$, $c_0, c_1, c_2, d \in \mathbb{R}$ and $A_0, A_1, B \in F_0^\perp$. Then, for $j \geq 0$ and all $t \in [t_0,t_1]$,

$$V_\delta^{(j)}(t) \ = \ \hat{V}_1^{(j)}(t) + O(\delta^2)$$

where the asymptotic constants depend only on d, t_0, t_1 and j. $\square$

For $n \geq 2$, integration of equation (4) gives

$$f_n^{(2)} \ = \ \sum_{i=1}^{n-1} \binom{n}{i} I([v_{n-i}^{(2)}, v_i]) \tag{8}$$

$$v_n^{(2)} \ = \ \sum_{i=1}^{n-1} \binom{n}{i} \tilde{i}\tilde{e}I(f_{n-i}^{(2)}\tilde{e}^{-1}v_i - f_i\tilde{e}^{-1}v_{n-i}^{(2)}) \tag{9}$$

where $\tilde{i} := \mathrm{ad}(F_0)$ and, for a continuous curve g of linear endomorpisms of $so(3)$, $I(g)(t) := \int_{t_0}^t g(s)\ ds$. Then $\tilde{i}$ commutes with $\tilde{e}$, and $\tilde{i}^2 = -\mathbf{1}$ where $\mathbf{1}$ is the identity on $so(3)$. If g is C^1, then

$$I(\tilde{e}g) \ = \ \frac{\tilde{i}}{d}(\tilde{e}g - g(t_0) - I(\tilde{e}g^{(1)})), \quad \text{in particular} \quad I(\tilde{e}) \ = \ \frac{\tilde{i}}{d}(\tilde{e} - \mathbf{1}). \tag{10}$$

Taking $n = 2$ in equation (8), and writing $A(t) = A_0 + (t - t_0)A_1$ with $A_0, A_1 \in so(3)$, repeated use of (10) gives

$$\begin{aligned} f_2 &= -2\langle [A_0, L_0(B)] + [A_1, L_1(B)], F_0\rangle \\ v_2 &= 2q^{(2)}(M_0(A_0) + M_1(A_1) - M_B(B)) + 2d^2\tilde{i} \circ I^2(I(q)\tilde{e})(B) \end{aligned}$$

where $L_0(t), L_1(t), M_0(t), M_1(t), M_B(t)$ are the endomorphisms of $so(3)$ given by

$$L_0 := \frac{1}{d}(-u\mathbf{1} + (u^2/2 - 1)\tilde{i} + \tilde{i} \circ \tilde{e}) \tag{11}$$

$$L_1 := \frac{1}{d^2}((u^2/2 - 3)\mathbf{1} + 2u\tilde{i} + 3\tilde{e} + u\tilde{i} \circ \tilde{e}) \tag{12}$$

$$M_0 := \frac{1}{d^3}((u^2/2 - 1)\mathbf{1} + u\tilde{i} + \tilde{e}) \tag{13}$$

$$M_1 := \frac{1}{d^4}((u^3/6 - u)\mathbf{1} + (u^2/2 - 1)\tilde{i} + \tilde{i} \circ \tilde{e}) \tag{14}$$

$$M_B := \frac{1}{d^3}(2(\tilde{e} - \mathbf{1}) + u\tilde{i} \circ (\tilde{e} + \mathbf{1})) \tag{15}$$

with $\tilde{e}$ evaluated at t, and $u := d(t - t_0)$. This proves

Theorem 3 *The second order approximate quadratic of V is*

$$\hat{V}_2(t) = D + \delta(q(t)F_0 + A_0 + (t - t_0)A_1 + \tilde{e}(t)(B)) + \frac{\delta^2}{2}(f_2(t)F_0 + v_2(t))$$

where q, A_0, A_1, B are as before. For $j \geq 0$ and all $t \in [t_0, t_1]$,

$$V_\delta^{(j)}(t) = \hat{V}_2^{(j)}(t) + O(\delta^3)$$

where the asymptotic constants depend only on d, t_0, t_1 and j. $\square$

Notice that q, A, B contain $3 + 4 + 2$ scalar variables, sufficient for initial conditions of the 3rd order ODE (3) in $so(3)$.

Example 9 *Using* ad *to identify $so(3)$ with E^3, Figure 1 plots (blue) a numerical solution of (2) for a Lie quadratic $V_\delta : [0, 5] \to so(3) \cong E^3$ near $V_0 = (1, 0, 0)$. The numerical solution is obtained using Mathematica's NDSolve, and V_δ is the non-null Lie quadratic specified by the initial conditions*

$$V_\delta(0) = (1.005, 0.006, -0.01), \quad V_\delta^{(1)}(0) = (-0.005, -0.00449, 0),$$

$$V_\delta^{(2)}(0) = (0.001, -0.005, 0.005), \quad V_\delta^{(3)}(0) = (0.00002, 0.005035, 0.005031)$$

with $C \approx (0.0009551, -0.00495, 0.00051755)$.

The approximations (green) $\hat{V}_1$ and (red) $\hat{V}_2$ are nearly indistinguishable from V_δ. The second degree[4] Taylor polynomial, also shown (dashed) in Figure 1, is a poor approximation to V_δ. The initial dot labels $V_\delta(0)$, and the second dots correspond to $t = 2$.

Figure 2 plots the same V_δ (blue), $\hat{V}_1$ (green) and $\hat{V}_2$ (red) for $t \in [0, 25]$. At first, both approximations successfully follow the contortions of V_δ, but as t increases $\hat{V}_1$ loses accuracy. The initial dot labels $V_\delta(0)$, the second dot corresponds to $V_\delta(2)$, and the arrows label V_δ, $\hat{V}_1$ and $\hat{V}_2$ at $t = 22.5$. The second order approximation $\hat{V}_2$ holds on longer, but eventually succumbs. By then, V_δ is far from the constant Lie quadratic V_0. $\square$

Theorems 2, 3 give explicit formulae in terms of elementary functions for approximations $\hat{V}_n$ to nearly constant Lie quadratics V_δ in E^3. Order n approximations x to Riemannian cubics x_δ in $SO(3)$ with Lie quadratic V_δ can be found by solving the linear system of ODEs

$$x^{(1)}(t) = x(t)\hat{V}_n(t) \tag{16}$$

[4] Higher order Taylor polynomials are almost as uncompetitive.

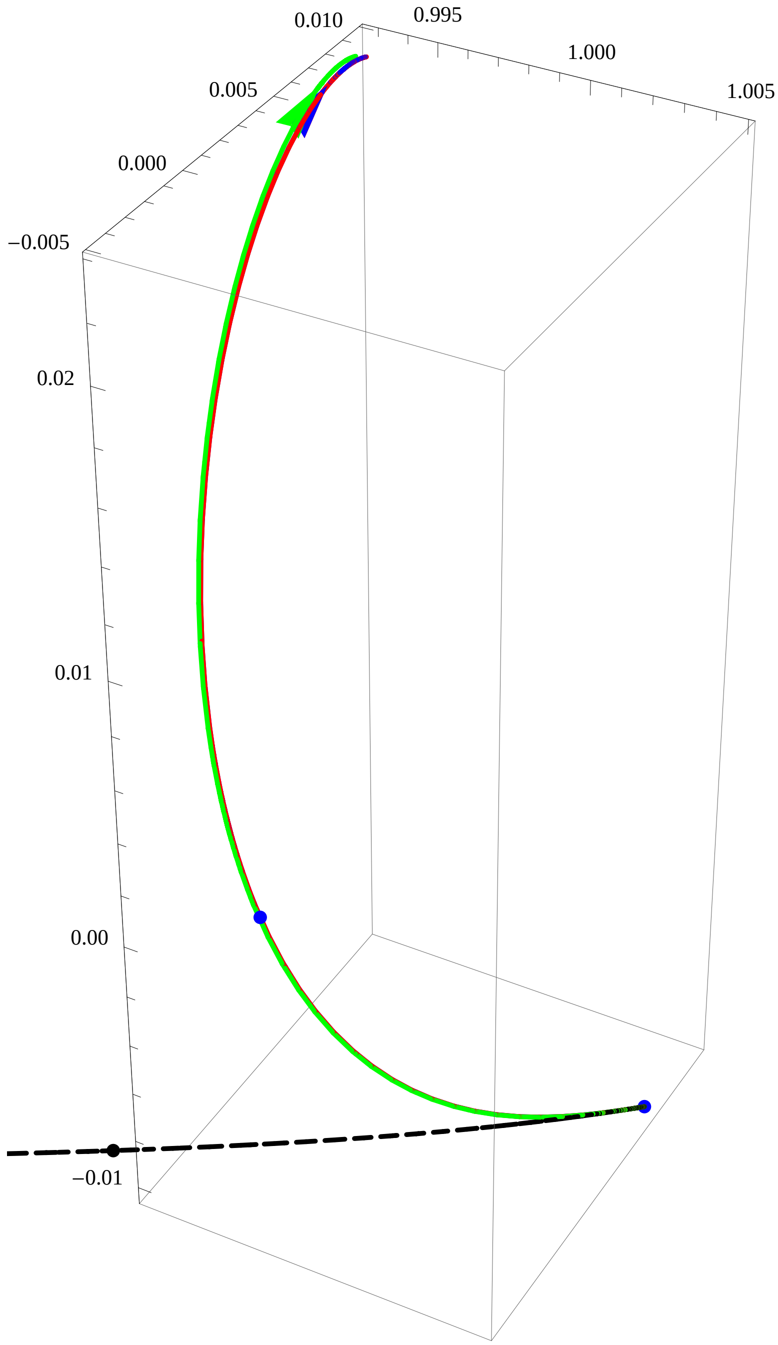

Figure 1. $V_\delta(t)$ (blue), $\hat{V}_1(t)$ (green), $\hat{V}_2(t)$ (red) and degree 2 Taylor (dashed) for $t \in [0, 5]$

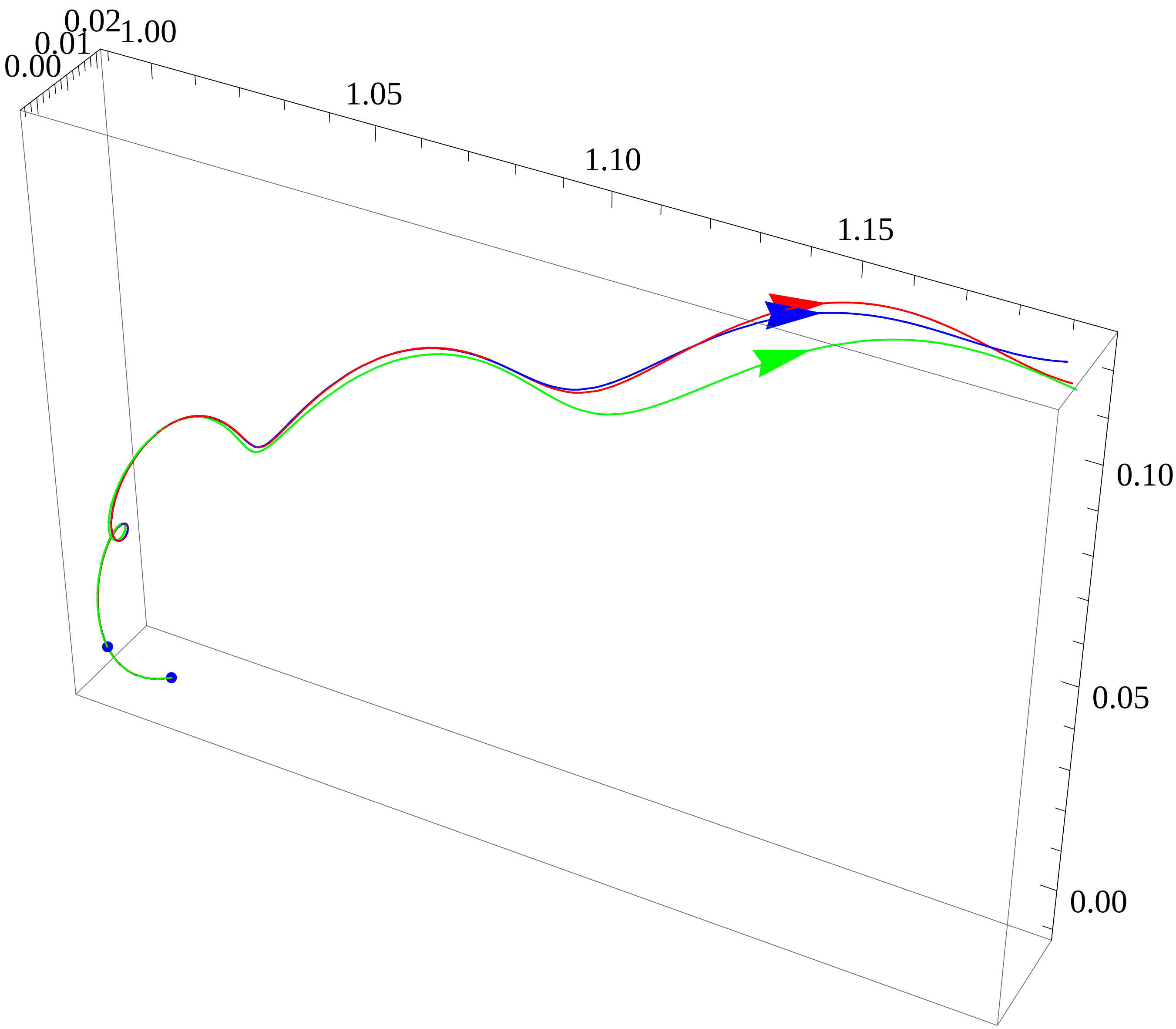

Figure 2. $V_\delta(t)$ (blue), $\hat{V}_1(t)$ (green), $\hat{V}_2(t)$ (red) for $t \in [0, 25]$

where, on the right hand side, $\hat{V}_n(t)$ is identified with an element of $so(3)$ and, considered as a 3×3 matrix, is premultiplied by the unknown matrix $x(t)$. Because (16) has nonconstant coefficients, x cannot be written down directly and would be found by a numerical integrator such as NDSolve. Considering that Riemannian cubics in $SO(3)$ are solutions of the 36 dimensional nonlinear system (1), the 9 dimensional linear system (16) appears comparatively benign. But solving (16) is nontrivial, and this step can be avoided: an explicit first order approximation $\hat{x}$ to x_δ is given as follows.

6 Approximating Nearly Geodesic Cubics in $SO(3)$

Given a generic Lie quadratic in a semisimple Lie algebra, there is an integrability algorithm [13] that takes a single quadrature, and gives an explicit formula for an associated Riemannian cubic. The algorithm takes a simple form [12] for Riemannian cubics in $SO(3)$. So we might try to approximate a nearly geodesic Riemannian cubic $x_\delta : [t_0, t_1] \to SO(3)$ to first order by substituting $\hat{V}_1$ for V_δ in the integrability algorithm. This has the unexpected benefit of giving a first order approximation for the quadrature in terms of elementary functions. On the other hand, $\hat{V}_2$ is needed to approximate the other terms in the integrability algorithm to first order. This gives, by somewhat indirect means, explicit first order approximations $\hat{x}$ to nearly geodesic Riemannian cubics x in $SO(3)$, without even the need for a single quadrature. To review the integrability algorithm for Lie quadratics we make two definitions.

Definition 2 *For $r \in \mathbb{R}$, let $R_r \in SO(3)$ be clockwise rotation by r in E^2, namely*

$$R_r \ := \ \begin{bmatrix} \cos r & \sin r & 0 \\ -\sin r & \cos r & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

□

Definition 3 *For linearly independent $X_1, X_2 \in E^3$, set*

$$S(X_1, X_2) \ := \ \begin{bmatrix} \frac{\|X_1\| X_2 - \langle X_1, X_2\rangle X_1}{\sqrt{\|X_1\|^2 \|X_2\|^2 - \langle X_1, X_2\rangle^2}} & \frac{X_1 \times X_2}{\sqrt{\|X_1\|^2 \|X_2\|^2 - \langle X_1, X_2\rangle^2}} & \frac{X_1}{\|X_1\|} \end{bmatrix}^{\mathbf{T}} \ \in \ SO(3)$$

where $^{\mathbf{T}}$ means transpose. Then, for any curve $W : [t_0, t_1] \to E^3$ with $W(t), W^{(1)}(t)$ everywhere linearly independent, define $T(W) : [t_0, t_1] \to SO(3)$ by

$$T(W)(t) \ := \ S(W(t), W^{(1)}(t)).$$

□

Now we return to Lie quadratics[5] V. As is easily proved [10], they have constant acceleration: $c := \langle V^{(2)}(t), V^{(2)}(t)\rangle$ is constant as t varies. When $V^{(3)}$ vanishes identically in a nonempty open interval, the corresponding Riemannian cubics are cubically reparameterised geodesics. Assume $V^{(3)}(t) \neq \mathbf{0}$ for any $t \in [t_0, t_1]$, where $V : [t_0, t_1] \to so(3) \cong E^3$, and E^3 is identified with $so(3)$ in the standard way, by the adjoint reprsentation.

Lemma 1 *Let $x : [t_0, t_1] \to SO(3)$ be a Riemannian cubic whose Lie quadratic is V. For $t \in [t_0, t]$, set*

$$y(t) \ := \ R_{\phi(t)} \circ T(V^{(2)})(t) \quad \text{where} \quad \phi(t) \ := \ c^{1/2} \int_{t_0}^{t} \frac{c - \langle C, V^{(2)}(s)\rangle}{\|V^{(3)}(s)\|^2} \, ds$$

and $\circ$ stands for matrix multiplication. Then $x(t) = x(t_0) y(t_0)^{\mathbf{T}} y(t)$.

Proof: Since left Lie reductions are invariant with respect to left multiplication, we can suppose without loss of generality that $x(t_0)$ is the identity. Setting $W_3(t) := c^{-1/2} V^{(2)}(t)$, $W_1(t) := V^{(3)}(t)/\|V^{(3)}(t)\|$, and $W_2(t) := W_3(t) \times W_1(t)$, we find that W_1, W_2, W_3 meet the requirements of Theorem 5 of [12] (in [12] a precise choice of W_1

[5] Here the symbol V is used to denote a Lie quadratic, rather than a variation of Lie quadratics.

is not made, allowing other curves in S^2 orthogonal to W_3). So there is a Riemannian cubic y of the form above, with Lie quadratic V, provided

$$\phi^{(1)}(t) \;=\; \langle W_1(t), W_2^{(1)}(t) + \mathrm{ad}^{-1}(V(t)) \times W_2(t)\rangle \;=\; -\,\langle W_1^{(1)}(t) + \mathrm{ad}^{-1}(V(t)) \times W_1(t), W_2(t)\rangle.$$

With the present choice of W_1, $\phi^{(1)} = \dfrac{1}{c^{1/2}\|V^{(3)}\|^2}\langle V^{(3)}, [V^{(2)}, V^{(4)}] + [V, [V^{(2)}, V^{(3)}]]\rangle =$

$$\frac{\langle V^{(3)}, [V^{(2)}, [V^{(3)}, V] + [V^{(2)}, V^{(1)}]] + [V, [V^{(2)}, V^{(3)}]]\rangle}{c^{1/2}\|V^{(3)}\|^2} \;=\; \frac{\langle V^{(3)}, [V^{(2)}, [V^{(2)}, V^{(1)}]]\rangle}{c^{1/2}\|V^{(3)}\|^2} \;=$$

$$-c^{1/2}\frac{\langle V^{(3)}, V^{(1)}\rangle}{\|V^{(3)}\|^2} \;=\; c^{1/2}\frac{\langle V^{(2)}, [V^{(1)}, V]\rangle}{\|V^{(3)}\|^2} \;=\; c^{1/2}\frac{(c - \langle C, V^{(2)}\rangle)}{\|V^{(3)}\|^2}.$$

□

So Riemannian cubics in $SO(3)$ can be found from Lie quadratics by a single quadrature. First order approximations to nearly geodesic Riemannian cubics can be written explicitly in terms of elementary functions, without the need for quadrature. More precisely, let $x_\delta : [t_0, t_1] \to SO(3)$ be a Riemannian cubic whose Lie quadratic $V_\delta : [t_0, t_1] \to so(3)$ is nearly constant. For $n = 1, 2$ let $\hat{V}_n : [t_0, t_1] \to so(3)$ be the order n approximations to V_δ given in §5. Assuming $B \neq \mathbf{0}$, write $B = \beta(\cos\gamma, \sin\gamma)$ where $\gamma \in [0, 2\pi)$. Set $\rho := -2c_2/(d^2\beta)$.

Theorem 4 *For $t \in t_0, t_1]$ define $\hat{x}(t) := x_\delta(t_0)\hat{y}(t_0)^{\mathbf{T}}\hat{y}(t)$ where $\hat{y}(t) := R_{\hat{\phi}(t)} \circ T(\hat{V}_2^{(2)})(t)$ and*

$$\hat{\phi}(t) \;:=\; \delta(\rho^2+1)^{1/2}((t-t_0)\beta + \frac{a_{11}(\cos(\gamma - d(t-t_0)) - \cos\gamma) + a_{12}(\sin(\gamma - d(t-t_0)) - \sin\gamma)}{d^2}).$$

Then for $j \geq 0$, and all $t \in [t_0, t_1]$, $x_\delta^{(j)}(t) = \hat{x}^{(j)}(t) + O(\delta^2)$, where the asymptotic constants depend on $a_{11}, a_{12}, \beta, c_2, d, t_0, t_1$ and j.

Proof: Let $\hat{W}_3(t)$ and $\hat{W}_1(t)$ be the unit vectors in the directions of $\hat{V}_2^{(2)}(t)$ and $\hat{V}_2^{(3)}(t)$ respectively. Taking $V = V_\delta$ in Lemma 1, it suffices to show that $\phi(t) = \hat{\phi}(t) + O(\delta^2)$ and that $W_i(t) = \hat{W}_i(t) + O(\delta^2)$ for $i = 3, 1$.

By Theorem 2, $V_\delta^{(2)} = \hat{V}_1^{(2)} + O(\delta^2) = \hat{V}_1^{(2)}(1 + O(\delta))$. So $c = \hat{c}(1 + O(\delta))$ and $C = \hat{C}(1 + O(\delta))$ where $\hat{c} := \|\hat{V}_1^{(2)}\|^2 = \delta^2(4c_2^2 + d^4\beta^2)$, $\hat{C} := \delta\ \mathrm{ad}(2c_2, -da_{12}, da_{11}) = \hat{V}_1^{(2)} - [\hat{V}_1^{(1)}, \hat{V}_1] + O(\delta^2)$. Similarly $V_\delta^{(3)} = \hat{V}_1^{(3)}(1 + O(\delta)) \Longrightarrow \|V_\delta^{(3)}\|^2 = d^6\beta^2 + O(\delta)$. So

$$\phi(t) \;=\; c^{1/2}\int_{t_0}^{t}\frac{c - \langle C, V_\delta^{(2)}(s)\rangle}{\|V_\delta^{(3)}(s)\|^2}\,ds \;=\; \hat{c}^{1/2}\int_{t_0}^{t}\frac{\hat{c} - \langle \hat{C}, \hat{V}_1^{(2)}(s)\rangle}{\|\hat{V}_1^{(3)}(s)\|^2}\,ds + O(\delta^2).$$

Because the denominator is constant, the integral on the right can be computed precisely, giving

$$\begin{aligned}\phi(t) \;&=\; \delta(4c_2^2 + d^4\beta^2)^{1/2}(\frac{t-t_0}{d^2} + \frac{a_{11}(\cos(\gamma - d(t-t_0)) - \cos\gamma) + a_{12}(\sin(\gamma - d(t-t_0)) - \sin\gamma)}{d^4\beta}) + O(\delta^2)\\ &=\; \hat{\phi}(t) + O(\delta^2).\end{aligned}$$

By Theorem 3, $V_\delta^{(2)} = \hat{V}_2^{(2)} + O(\delta^3) = \hat{V}_2^{(2)}(1 + O(\delta^2))$. So $W_3 = \hat{W}_3 + O(\delta^2)$. Similarly $W_1 = \hat{W}_1 + O(\delta^2)$. □

Whereas in [11] the analysis of Riemannian cubics in $SO(3)$ is complicated, in the present paper $\hat{x}$ is algebraic in terms of polynomials and trigonometric functions, as we see by taking $n = 2$ in equations (8), (9):

$$\begin{aligned}f_2^{(2)} \;&=\; 2d[A_0, \tilde{i}(\tilde{e} - \mathbf{1})B] + [A_1, 2(\tilde{e} - \mathbf{1} + d(t-t_0)\tilde{i}\tilde{e})B]\\ v_2^{(2)} \;&=\; -\frac{4c_2}{d}(\tilde{e} - \mathbf{1})A_0 + \frac{4c_2}{d^2}(d(t-t_0)\mathbf{1} - \tilde{i}(\tilde{e} - \mathbf{1}))A_1 + 2(2c_2(t-t_0) + d^2 I(q))\tilde{i}\tilde{e}B\end{aligned}$$

where everything is constant, except t,

$$\tilde{e}(t) \;=\; \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(d(t-t_0)) & \sin(d(t-t_0)) \\ 0 & -\sin(d(t-t_0)) & \cos(d(t-t_0)) \end{bmatrix} \quad \text{and} \quad I(q)(t) \;=\; \frac{c_2(t-t_0)^3}{3} + \frac{c_1(t-t_0)^2}{2} + c_0(t-t_0).$$

Then $\hat{V}_2^{(2)} = \delta(2c_2F_0 - d^2\tilde{e}B) + \frac{\delta^2}{2}(f_2^{(2)}F_0 + v_2^{(2)})$ is affine in $\tilde{e}(t)B$, $f_2^{(2)}(t)$ and $v_2^{(2)}(t)$. So in Theorem 4, $T(\hat{V}_2^{(2)})$ is $S(X_1, X_2)$ where

$$\begin{aligned} X_1(t) &:= 2c_2F_0 - d^2\tilde{e}(t)B + \frac{\delta}{2}(f_2^{(2)}(t)F_0 + v_2^{(2)}(t)) \quad \text{and} \\ X_2(t) &:= d^3\tilde{i}\tilde{e}(t)B + \frac{\delta}{2}(f_2^{(3)}(t) + v_2^{(3)}(t)). \end{aligned}$$

Example 10 *For some small $\delta \in \mathbb{R}$, let $V_\delta : [0,10] \to so(3)$ be the Lie quadratic satisfying*

$$\mathrm{ad}(V_\delta(0)) \;=\; (1,0,0) + \delta(0,1,0), \quad \mathrm{ad}(V_\delta^{(1)}(0)) \;=\; \delta(0,0,1)/2, \quad \mathrm{ad}(V_\delta(2)(0)) \;=\; \delta(1,1,1)/4 \tag{17}$$

and let $x_\delta : [0,8] \to SO(3)$ be the corresponding Riemannian cubic for which $x_\delta(0) = \mathbf{1}$.

Given any particular value of δ, say $\delta = 0.05$, Mathematica's NDSolve can be used to numerically solve the quadratic differential equation (1) for $V_\delta : [0,8] \to so(3)$. Then x_δ is found by numerically solving the linear differential equation

$$x_\delta^{(1)}(t) \;=\; x_\delta(t)V_\delta(t)$$

where the coefficients on the right are entries of the matrix $V_\delta(t)$. The second rows of $x_\delta(t)$ are shown as the blue curve in Figure 3, with blue labelled points corresponding to $t = 0, 1, 2, \ldots 10$. A geodesic represented in this way would appear as a circular arc.

To compute the approximate solution $\hat{x} : [0,10] \to SO(3)$ from Theorem 4, take $t_0 = 0, t_1 = 10$ and $D = (1,0,0)$. For $\hat{x}$ with $\hat{x}(0) = \mathbf{1}$ to satisfy the other initial conditions (17) with $O(\delta^2)$ errors, it suffices to have

$$\hat{V}_1(0)) \;=\; (1,\delta,0), \quad \hat{V}_1^{(1)}(0) \;=\; (0,0,\delta/2), \quad \hat{V}_1(2)(0) \;=\; \delta(1,1,1)/4$$

where $\hat{V}_1$ is considered as a curve in E^3. For this, choose parameters $c_0, c_1, c_2, a_{01}, a_{02}, a_{11}, a_{12}, \beta, \gamma$ so that

$$\begin{aligned} (c_0, a_{01} + \beta\cos\gamma, a_{02} + \beta\sin\gamma) &= (0,1,0) \\ (c_1, a_{11} + \beta\sin\gamma, a_{12} - \beta\cos\gamma) &= (0,0,1)/2 \\ (c_2, -\beta\cos\gamma, -\beta\sin\gamma) &= (1,1,1)/4 \end{aligned}$$

Taking $\beta = \sqrt{2}/4$ we have $c_0 = c_1 = 0$, $c_2 = 1/8$, $\gamma = 5\pi/4$, $a_{01} = 5/4$, $a_{02} = 1/4$, $a_{11} = a_{12} = 1/4$. The second rows of $\hat{x}(t)$ are shown as the red curve in Figure 3, with red points corresponding to $t = 3, 4, 5, \ldots 10$. For $t = 0, 1, 2$, $\hat{x}(t)$ and $x_\delta(t)$ are hard to distinguish. For $t = 3, 4, 5$ the approximation is good enough for the blue numbers to label both.

It is hard to see much difference between $\hat{x}(t)$ and $x_\delta(t)$ until around $t = 4$, and long before then the Riemannian cubic x_δ is obviously nongeodesic. After that, differences between $\hat{x}(t)$ and $x_\delta(t)$ become noticeable ($t = 5$), and then large ($t \geq 6$) where x_δ is very far from geodesic. □

7 Conclusion

We derive first order approximations $\hat{x}$ to nearly geodesic Riemannian cubics x_δ in $SO(3)$ defined over a given interval $[t_0, t_1]$. The approximations capture a good deal of the geometry of x_δ, including behaviour of the associated derivative, corresponding to the body angular velocity for a rigid body. Equivalently, the approximations apply to restrictions of arbitrary Lie quadratics to sufficiently small intervals. This fills a gap in the literature, by complementing studies on long term asymptotics of Riemannian cubics.

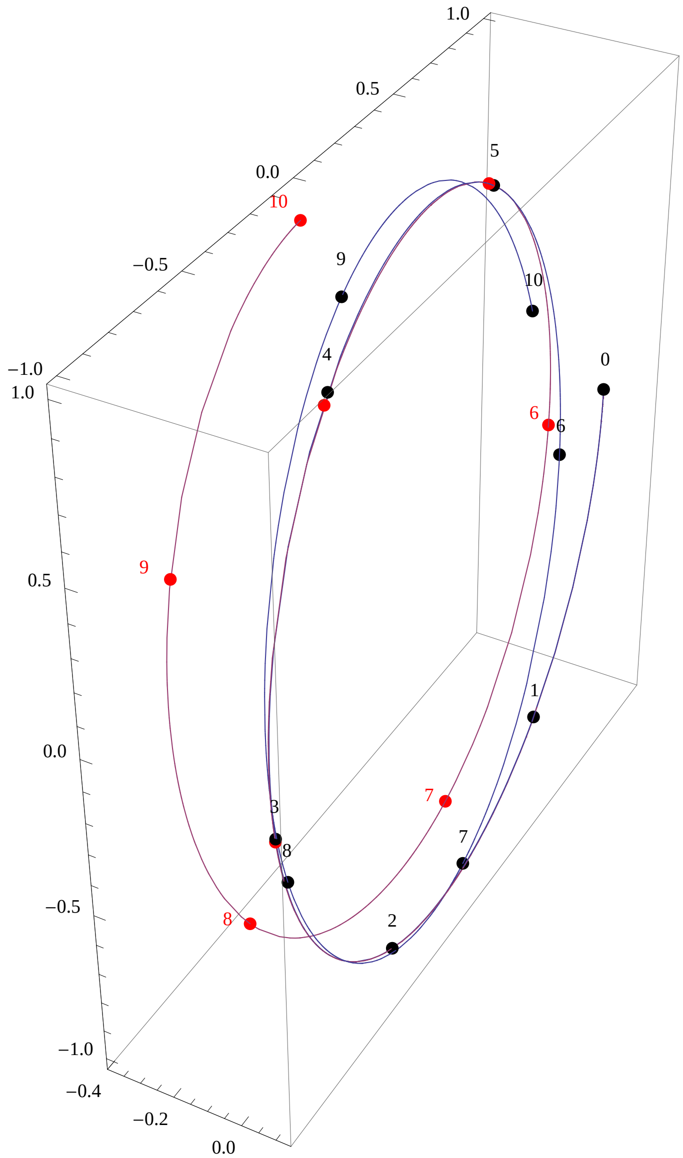

Figure 3. Approximation (second rows) to a Nearly Geodesic Riemannian Cubic in $SO(3)$

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